A NEW THREE STEP SECOND DERIVATIVE LINEAR MULTISTEP METHODS FOR THE SOLUTIONS OF STIFF INITIAL VALUE PROBLEMS

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Abstract: This paper is concerned with the accuracy and efficiency of a three step second derivative linear multistep method for the approximate solution of stiff initial value problems. The main methods were derived by blending of two linear multistep methods using continuous collocation approach. These methods are of uniform order four. The stability analysis of the block methods indicates that the methods are A–stable, consistent and zero stable hence convergent. Numerical results obtained using the proposed new block methods were compared with those obtained by the well known ODE solver ODE15S to illustrate its accuracy and effectiveness. The proposed block method is found to be efficient and accurate hence recommended for the solution of stiff initial value problems.

Keywords: Blended block, linear multistep methods, stiff ODEs, continuous collocation

Introduction

In this research paper, the construction and application of a three step order four blended block linear multistep method for the numerical solutions of stiff initial value problems (1) was considered. A potentially good numerical method for the solution of stiff system of ordinary differential equations (ODEs) must have good accuracy and some wide region of absolute stability. One of the first and most important stability requirements for linear multistep methods is A-stability as proposed by Enright (1974). The three step blended block linear multistep methods is A-stable hence the application of the method here which makes it suitable for the solution of linear and non linear ODEs.

The solution of stiff system of ODEs has been considered by Chollom *et al.* (2011) where a block hybrid Adams Moulton Method was used (Kumleng *et al.,* 2013) and where ten step block generalized Adams method was used. Many has discussed the solution of linear and non linear ODEs from different basis functions, among them are Onumanyi *et al* (1994), Sirisena *et al.* (2004), Kumleng (2012) and so on.

The three step blended linear multistep method

The three step blended linear multistep method is constructed based on the continuous finite difference approximation approach using the interpolation and collocation criteria described by Lie and Norsett (1981) called multistep collocation (MC) and block multistep methods by Onumanyi *et al.* (1994, 1999). We define based on the interpolation and collocation methods the continuous form of the k- step second derivative new method as;

$$y(x) = \sum_{j=1}^{1} a_j(x) y_{n+j} + h \sum_{j=0}^{m-1} b_j(x) f_{n+j} + h^2 l_k(x) y''_{n+k}$$

$$a_{k-1}(x) = \sum_{i=0}^{t+m-1} a_{j,i+1} x^i \quad j = 0,1,\ldots,t-1$$

$$b_j(x) = \sum_{i=0}^{t+m-1} b_{j,i+1} x^i, \; j = 0,1,2,\ldots,m-1$$

and

$$l_k(x) = \sum_{i=0}^{t+m-1} l_{k,i+1} x^i, \; j = 0,1,2,\ldots,m-1 \tag{6}$$

are the continuous coefficients of the method, m is the number of distinct collocation points, h is the step size and from Onumanyi *et al.* (1994), we obtain our matrices D and $C = D^{-1}$ by the imposed conditions expressed as DC = I; **where:**

$$D=\begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+k-1} & x_{n+k-1}^2 & \cdots & x_{n+k-1}^{t+m-1} \\ 0 & 1 & 2\bar{x}_0 & \cdots & (t+m-1)\bar{x}_0^{\,t+m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2\bar{x}_{m-1} & \cdots & (t+m-1)\bar{x}_{m-1}^{\,t+m-2} \\ 0 & 0 & 2 & \cdots & (t+m-2)(t+m-1)\bar{x}_0^{\,t+m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 2 & \cdots & (t+m-2)(t+m-1)\bar{x}_{m-1}^{\,t+m-3} \end{bmatrix} \tag{7}$$

$$C=\begin{pmatrix} \alpha_{01} & \alpha_{11} & \cdots & \alpha_{t-1,1} & h\beta_{01} & h\lambda_{01} & \cdots & h\lambda_{m-1,1} \\ \alpha_{02} & \alpha_{12} & \cdots & \alpha_{t-1,2} & h\beta_{02} & h\lambda_{02} & \cdots & h\lambda_{m-1,2} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & h\lambda_{0,t+m} & \cdots & h\lambda_{m-1,t+m} \end{pmatrix} = D^{-1} \tag{8}$$

respectively.

In this case, k=3,t=1 and m=5 and it continuous form expressed in the form of (6) is

$$y(x) = \alpha_2(x)y_{n+2} + h\sum_{j=0}^{m-1} \beta_j(x) f_{n+j} + h^2\lambda_3(x)y''_{n+3} \tag{9}$$

$$\tilde{y}(x_{n+9}) = \alpha_2(x)y_{n+2} + h\left[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3}\right] + h^2\lambda_3(x)y''_{n+3} \tag{10}$$

Using the approach of Onumanyi *et al.* (1999); the matrix form of

$$D=\begin{bmatrix} 1 & (x_n+2h) & (x_n+2h)^2 & (x_n+2h)^3 & (x_n+2h)^4 & (x_n+2h)^5 \\ 0 & 1 & 2x_n & 3{x_n}^2 & 4{x_n}^3 & 5{x_n}^4 \\ 0 & 1 & 2(x_n+h) & 3(x_n+h)^2 & 4(x_n+h)^3 & 5(x_n+h)^4 \\ 0 & 1 & 2(x_n+2h) & 3(x_n+2h)^2 & 4(x_n+2h)^3 & 5(x_n+2h)^4 \\ 0 & 1 & 2(x_n+3h) & 3(x_n+3h)^2 & 4(x_n+3h)^3 & 5(x_n+3h)^4 \\ 0 & 0 & 2 & 6(x_n+3h) & 12(x_n+3h)^2 & 20(x_n+3h)^3 \end{bmatrix} \tag{11}$$

Using the Maple software, the inverse of the matrix in (11) is obtained and its elements are used in obtaining the continuous coefficients and substituting these continuous coefficients into (9) yields the continuous form of our new method. The continuous form as:

$$\begin{aligned} \tilde{y}(\tau+x_n) &= y_{n+2} + \left(\tau - \frac{13\tau^2}{12h} + \frac{29\tau^3}{54h^2} - \frac{\tau^4}{8h^3} + \frac{\tau^5}{8h^4} - \frac{43h}{135}\right) f_n \\ &+ \left(\frac{9\tau^2}{4h} - \frac{7\tau^3}{4h^2} + \frac{\tau^4}{2h^3} - \frac{\tau^5}{20h^4} - \frac{7h}{5}\right) f_{n+1} \\ &+ \left(-\frac{9\tau^2}{4h} + \frac{5\tau^3}{2h^2} - \frac{7\tau^4}{8h^3} + \frac{\tau^5}{10h^4} - \frac{h}{5}\right) f_{n+2} \\ &+ \left(\frac{13\tau^2}{12h} - \frac{139\tau^3}{108h^2} + \frac{\tau^4}{2h^3} + \frac{11\tau^5}{180h^4} - \frac{11h}{135}\right) f_{n+3} \\ &+ \left(\frac{-\tau^2}{2} + \frac{11\tau^3}{18h} - \frac{\tau^4}{4h^2} + \frac{\tau^5}{30h^3} - \frac{2h}{45}\right) g_{n+3} \end{aligned} \tag{12}$$

$g_{n+3} = y''_{n+3}$

Evaluating the continuous scheme (12) at $t = 0, h, 3h$ gives the three discrete methods which constitute the three step blended block linear multistep method.

$$\begin{aligned} y_n &= y_{n+2} - \frac{1}{135}\{43hf_n + 189hf_{n+1} + 27hf_{n+2}11hf_{n+3} \\ &\quad - 3h^2y''_{n+3}\} \\ y_{n+1} &= y_{n+2} - \frac{1}{1080}\{23hf_n - 486hf_{n+1} - 783hf_{n+2} \\ &\quad + 249f_{n+3} - 66h^2y''_{n+3}\} \\ y_{n+3} &= y_{n+2} + \frac{1}{1080}\{7hf_n - 54hf_{n+1} + 513hf_{n+2} + \\ &\quad 614hf_{n+3} - 114\,h^2y''_{n+3}\} \end{aligned} \tag{13}$$

Stability analysis of the new methods

In this section, we consider the analysis of the newly constructed methods. Their convergence is determined and their regions of absolute stability plotted.

Convergence

The convergence of the new block methods is determined using the approach by Fatunla (1991) and Chollom*et.al* (2007) for linear multistep methods, where the block methods are represented in a single block, r point multistep method of the form

$$\rho(r) = \sum_{i=0}^{k} \alpha_j r^j \qquad 14$$

Zero Stability of the BBLMM for k=3

To determine the zero stability of the BBLMM we use the approach of Ehigie (2007) for linear multistep methods where he expressed the methods in the matrix form as shown below. Following the work of Ehigie and Okunuga (2014); we observed that the seven step block method is zero stable as the roots of the equation

$$det\left(r\left(A - Cz - D1z^2\right) - B\right) = 0$$

are less than or equal to 1. Since the block method is consistent and zero-stable, the method is convergent (Henrici, 1962).

These new methods are consistent since their orders are 4, it is also zero-stable, above all, there are A – stable as can be seen in Fig. 1. The new three step discrete methods that constitute the block method have the following orders and error constants as shown below.

The three step blended block multistep methods has uniform order of $(4, 4, 4)^T$ and error constants of $C_6 = \left(\frac{-899}{100000}, \frac{458}{1000000}, \frac{-236}{1000000}\right)^T$

Regions of absolute stability of the methods

The absolute stability regions of the newly constructed blended block linear multistep methods (12) was plotted using Ehigie (2007).

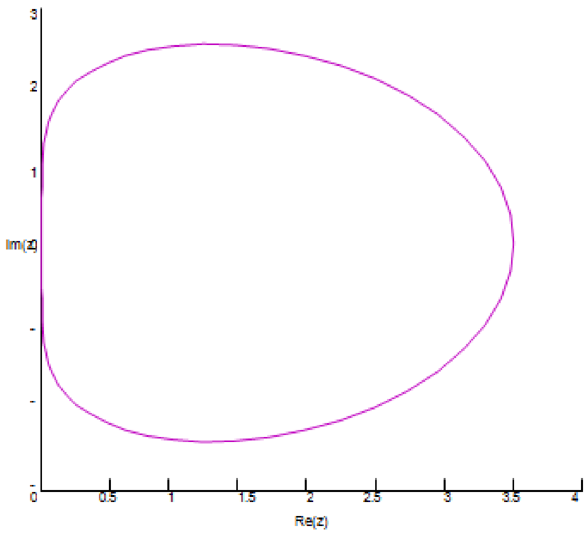

Fig 1: Absolute stability region For BBLMM For K=3

This absolute stability region is A –stable since it consist of the set of points in the complex plane outside the enclosed figure.

Numerical examples

We report here a numerical example on stiff problem taken from the literature using the solution curve. In comparison, we also report the performance of the new blended block linear multistep methods and the well-known Matlab stiff ODE solver ODE15S on the same problems and on the same axes.

Problem 1: Irregular Heartbeat and Lidocaine Model

The irregular heartbeat and Lidocaine model is expressed mathematically by the following ivp

$y_1' = -0.09\, y_1 + 0.038y_2$

$y_2' = 0.066\, y_1 - 0.038y_2$

$y_1(0) = y_2(0) = y_0$

$y_0 = \text{Maximum Safe Dosage} = 3\text{mw/kg}^3$

$0 \le x \le 700, \quad h = 0.1$

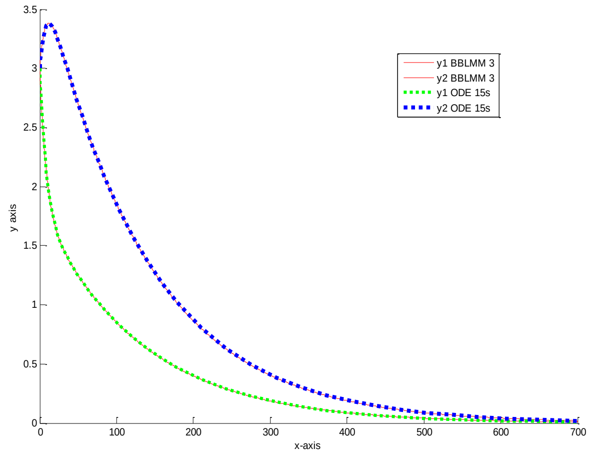

Fig. 2: Solution curves of Problem 1 solved by the our new methods

Problem 2: Van der Pol's Equations

$y_1' = y_2$

$y_2' = -y_1 - \mu y_2(1 - y_1^2)$

$\mu = 300, y_1(0) = 2, y_2(0) = 0\ 0 \le x \le 40, \quad h = 0.1$

Stiffness ratio3×10^2.

The Van der Pol's Equation is an important kind of second-order non-linear auto-oscillatory equation (Fig. 3). It is a non-conservative oscillator with non-linear damping.

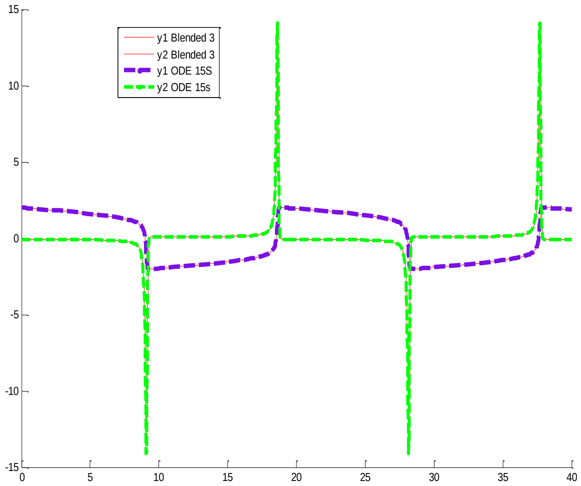

Fig. 3: Solution curves of Problem 2 solved by the our new methods

Conclusion

Problem 1 is a model of the relationship between Lidocaine and Irregular Heart beat. Lidocaine belong to a group of drugs known as anti-arrhythmic which work by preventing sodium from being pumped out on the cells of the heart to help the heart beat normally. From our solution curves, it was observed that normalcy in the heart beat can be attained with the use of Lidocaine within the correct dosage. Our solution curves coincide with the solutions of ODE 15s.

Van der Pol's equation in Problem 2 is a non conservative oscillator with non linear damping energy dissipated at high amplitude. From the solution curves the trajectories traces the motion of a single point through a flow with a limit circle where the trajectories spiral into or away from the limit circle. Our solution curves compete favourably with ODE 15s.

It can be seen clearly from the curve that our new methods perform favourably better than the well known ODE15S for the problems solved in Problems 1 and 2. It was also observed that the new methods have better stability regions than the conventional Adams Moulton method for step number 3.

Conflict of Interest

Authors declare that there is no conflict of interest reported on this work.

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